Characterizing Trees for Lambda-mu Terms

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Abstract. We give the conditions to characterize Böhm tree structures which represent terms of the Lambda-mu calculus. This result answers a question stated in Saurin's FLOPS paper.

1 Introduction

In [3], Saurin extends the Böhm trees and the expanded Böhm trees, called Nakajima trees in [3], to the $\Lambda\mu$-terms. It is a beautiful theoretical result that we can obtain the (expanded) Böhm trees for $\Lambda\mu$ just by extending the bound of lengths of prefixes and arguments to ω^2, whereas it is ω for the λ-calculus. An open problem stated in [3] is on characterization of the tree structures which represent some $\Lambda\mu$-terms. For the λ-calculus, this problem was solved by Nakajima in [1], where he gave some conditions for the expanded Böhm trees and showed that they give the exact characterization of trees which represents some λ-terms.

In this paper, we extended Nakajima's result to $\Lambda\mu$, and solve Saurin's open problem. The idea of the characterizing conditions are almost the same as [1], and they require that the information of each node is computable, and that all of nodes except for a finite part are obtained by η-expansion. The conditions for $\Lambda\mu$-calculus become much more complicated than the case of λ-calculus, because we have to manage the correspondence of μ-variables in prefixes and bodies.

2 Expanded Böhm Trees of $\Lambda\mu$-Terms

Definition 1. The $\Lambda\mu$-terms are defined as follows:

$$t, s ::= x \mid \lambda x.t \mid (t)s \mid \mu\alpha.t \mid (t)\alpha$$

$\Sigma_{\Lambda\mu}$ is the set of all $\Lambda\mu$-terms. $\Sigma^c_{\Lambda\mu}$ is the set of all closed $\Lambda\mu$-terms. In this paper, we always suppose every $\Lambda\mu$-term is closed with respect to μ-variables.

The reduction rules for the $\Lambda\mu$-calculus are the following.

$$\begin{aligned}
(\lambda x.t)s &\to_{\beta_T} t[x := s] \\
(\mu\alpha.t)\beta &\to_{\beta_S} t[\alpha := \beta] \\
\mu\alpha.(t)\alpha &\to_{\eta_S} t && (\alpha \notin FV(t)) \\
\mu\alpha.t &\to_{\mathrm{fst}} \lambda x.\mu\alpha.t[(v)\alpha := (v)x\alpha]
\end{aligned}$$

As in [3], the stream head normal form (shnf) of a $\Lambda\mu$-term is

$$\lambda \boldsymbol{x}^0 \mu\alpha^0 \cdots \lambda \boldsymbol{x}^{n-1} \mu\alpha^{n-1}.(y)\boldsymbol{t}^0\beta^0 \cdots \boldsymbol{t}^{m-1}\beta^{m-1},$$

where each $\boldsymbol{x}^i$ and $\boldsymbol{t}^j$ are finite sequences of λ-variables and $\Lambda\mu$-terms. For simplicity, we write $t \to_h^* h$ for the head reduction followed by zero- or one-step η_S-reduction to a shnf h.

Saurin showed in [3] that the Böhm trees are adapted to the $\Lambda\mu$-calculus by extending the width from ω to ω^2. We consider fully η-expanded form, and hence each node uniformly has ω^2 children. We call such trees *expanded Böhm trees* for the $\Lambda\mu$-calculus, which are originally defined by coinduction as follows:

$$\mathfrak{T} ::= \bot \mid \Lambda(x_i)_{i\in\omega^2}.(y)(\mathfrak{T}_j)_{j\in\omega^2},$$

where $\Lambda(x_i)_{i\in\omega^2}.(y)$ is called a prefix. Positions of nodes in trees are expressed by finite lists of elements of ω^2.

Definition 2. Δ is the set of all finite lists consisting of elements of ω^2, that is, $[\,] \in \Delta$ (empty list), and if $\delta \in \Delta$ and $\mu \in \omega^2$, then $\delta :: \mu \in \Delta$. $\delta \leq \delta'$ means that δ is an initial segment of δ'. $\delta < \delta'$ means $\delta \leq \delta'$ and $\delta \neq \delta'$.

By renaming bound variables, we suppose that bound λ-variables in prefixes are uniformly indexed by an element of $\Delta \times \omega^2$ depending on the position where they are abstracted, that is, we define the set of all bound λ variables as $BV_\lambda = \{x_\delta^\mu \mid \delta \in \Delta, \mu \in \omega^2\}$, and the prefix at the position δ is fixed as $\lambda \boldsymbol{x}_\delta^0 \boldsymbol{x}_\delta^1 \cdots\cdots .(y)$ with some head variable y, where $\boldsymbol{x}_\delta^i = x_\delta^{\omega\cdot i} x_\delta^{\omega\cdot i+1} x_\delta^{\omega\cdot i+2} \cdots$. We use this notation $\boldsymbol{x}_\delta^i$ in the following, and another notation $\boldsymbol{x}_\delta^{i,<j} = x_\delta^{\omega\cdot i} x_\delta^{\omega\cdot i+1} \cdots x_\delta^{\omega\cdot i+j-1}$.

In order to consider the fst-reduction, we have to remember some information on bound μ-variables during the definition of the expanded Böhm trees of $\Lambda\mu$-terms. In the following definition, $\phi(\delta, k) = l$ means that the prefix of the shnf at the position δ contains a subterm of the form $\cdots \lambda \boldsymbol{x}_\delta^{k,<l} \mu\alpha_\delta^k \cdots$. Similarly to BV_λ, we fix the name of bound μ-variables depending on the position where they are abstracted. The set of bound μ-variables is $BV_\mu = \{\alpha_\delta^k \mid \delta \in \Delta, k \in \omega\}$.

Then, names of head variables at each position and existence of shnf are sufficient to characterize expanded Böhm trees.

Definition 3. An *expanded Böhm tree* for $\Lambda\mu$-calculus is a mapping $\mathfrak{T}$ from Δ to $BV_\lambda \cup \{\bot\}$ such that $\mathfrak{T}(\delta) = \bot$ and $\delta' > \delta$ imply $\mathfrak{T}(\delta') = \bot$. The set of the expanded Böhm trees is denoted by $\Lambda\mu$-$\mathfrak{BT}^+$. We write $\mathfrak{T}(\delta) \uparrow$ to mean $\mathfrak{T}(\delta) = \bot$, and $\mathfrak{T}(\delta) \downarrow$ otherwise.

We can intuitively understand this definition as follows: $\mathfrak{T}(\delta) = x_{\delta'}^\mu$ means that the head variable in the prefix at δ is the μ-th variable in the prefix at δ', and $\mathfrak{T}(\delta) \uparrow$ means that the node at δ is $\bot$, which represents an unsolvable term, and we suppose that all nodes below $\bot$ are indexed by $\bot$ for simplicity.

Definition 4. For $t \in \Sigma_{\Lambda\mu}^c$, we define the *expanded Böhm tree* $\mathfrak{BT}_t^+$ of t with auxiliary partial mappings $t_{(\cdot)} : \Delta \to \Sigma_{\Lambda\mu} \cup \{\bot\}$ and $\phi_t : \Delta \times \omega \longrightarrow \omega$ as follows:

(0) $\mathfrak{T}$ is recursive, and there exist the following five partial recursive mappings:

- $\mathsf{p}_\mu^{\mathfrak{T}}, \mathsf{b}_\mu^{\mathfrak{T}} : \Delta \longrightarrow \omega$, the domains of which are $\{\delta \in \Delta \mid \mathfrak{T}(\delta)\downarrow\}$,
- $\mathsf{p}_\lambda^{\mathfrak{T}}, \mathsf{b}_\lambda^{\mathfrak{T}} : \Delta \times \omega \longrightarrow \omega$, and $\mathsf{Bd}_\mu^{\mathfrak{T}} : \Delta \times \omega \longrightarrow \Delta \times \omega$, the domains of which are $\{\langle \delta, k\rangle \in \Delta \times \omega \mid \mathfrak{T}(\delta)\downarrow\}$.

(1) $\mathfrak{T}(\delta) = x_{\delta'}^{\mu} \Longrightarrow \delta' \le \delta$ and $\mathsf{Bd}_\mu^{\mathfrak{T}}(\delta, k) = \langle \delta', k'\rangle \Longrightarrow \delta' \le \delta$
(2) $\mathfrak{T}(\delta :: (\omega \cdot k + l)) = x_{\delta'}^{\omega \cdot k' + l'}$ & $l < \mathsf{b}_\lambda^{\mathfrak{T}}(\delta, k) \Longrightarrow l' < \mathsf{p}_\lambda^{\mathfrak{T}}(\delta', k')$
(3) $\mathsf{Bd}_\mu^{\mathfrak{T}}(\delta, k) = \langle \delta', k'\rangle$ & $k < \mathsf{b}_\mu^{\mathfrak{T}}(\delta) \Longrightarrow k' < \mathsf{p}_\mu^{\mathfrak{T}}(\delta')$
(4) $\mathsf{Bd}_\mu^{\mathfrak{T}}(\delta, k) = \langle \delta', k'\rangle$ & $l \ge \mathsf{b}_\lambda^{\mathfrak{T}}(\delta, k) \Longrightarrow$

- $\mathfrak{T}(\delta :: (\omega \cdot k + l)) = x_{\delta'}^{\omega \cdot k' + (\mathsf{p}_\lambda^{\mathfrak{T}}(\delta', k') + l - \mathsf{b}_\lambda^{\mathfrak{T}}(\delta, k))}$
- $\delta'' \ge \delta :: (\omega \cdot k + l) \Longrightarrow \mathfrak{T}(\delta'' :: \mu) = x_{\delta''}^{\mu}$ for any μ & $\mathsf{p}_\mu^{\mathfrak{T}}(\delta'') = 0$ & $\mathsf{b}_\mu^{\mathfrak{T}}(\delta'') = 0$

(5) $\mathsf{p}_\lambda^{\mathfrak{T}}(\delta, \mathsf{p}_\mu^{\mathfrak{T}}(\delta) + n) = 0$ and $\mathsf{b}_\lambda^{\mathfrak{T}}(\delta, \mathsf{b}_\mu^{\mathfrak{T}}(\delta) + n) = 0$ for any $n \in \omega$
(6) $\mathsf{Bd}_\mu^{\mathfrak{T}}(\delta, \mathsf{b}_\mu^{\mathfrak{T}}(\delta) + n) = \langle \delta, \mathsf{p}_\mu^{\mathfrak{T}}(\delta) + n\rangle$ for any $n \in \omega$

Fig. 1. Characterizing Conditions

- $t_{[]} = t$
- If t_δ has no shnf, $\mathfrak{B}\mathfrak{T}_t^+(\delta) = \bot$, and $t_{\delta'}$ for any $\delta < \delta'$ and $\phi_t(\delta'', k)$ for any $\delta \le \delta''$ and $k \in \omega$ are undefined.
- If $t_\delta \to_h^* \lambda \boldsymbol{x}_\delta^{0,<i_0} \mu\alpha_\delta^0 \cdots \lambda \boldsymbol{x}_\delta^{n-1,<i_{n-1}} \mu\alpha_\delta^{n-1}.(y)\boldsymbol{t}^0 \alpha_{\delta_0}^{j_0} \cdots \boldsymbol{t}^{m-1} \alpha_{\delta_{m-1}}^{j_{m-1}}$, then

$$\mathfrak{B}\mathfrak{T}_t^+(\delta) = y$$

$$\phi_t(\delta, k) = \begin{cases} i_k & (k < n) \\ 0 & (k \ge n) \end{cases}$$

$$t_{\delta :: (\omega \cdot k + l)} = \begin{cases} t_k^l & (k < m \;\&\; \boldsymbol{t}_k = t_k^0 \cdots t_k^{i_k - 1} \;\&\; l < i_k) \\ x_{\delta_k}^{\omega \cdot j_k + (\phi_t(\delta_k, j_k) + l - i_k)} & (k < m \;\&\; \boldsymbol{t}_k = t_k^0 \cdots t_k^{i_k - 1} \;\&\; l \ge i_k) \\ x_\delta^{\omega \cdot (k - m + n) + l} & (k \ge m) \end{cases}$$

Note that $\delta_k \le \delta$ holds for $0 \le k < m$ since $\alpha_{\delta_k}^{j_k}$ is a bound μ-variable in t.

3 Characterization

3.1 Characterization of Expanded Böhm Trees for $\Lambda\mu$-Terms

For each $\mathfrak{T} \in \Lambda\mu$-$\mathfrak{B}\mathfrak{T}^+$, we consider the conditions in Figure 1. The intuitive meaning of the partial mappings is the following: $\mathsf{p}_\mu^{\mathfrak{T}}(\delta)$ is the number of μ-abstractions in the prefix of the shnf of t_δ. $\mathsf{p}_\lambda^{\mathfrak{T}}(\delta, k)$ is the number of λ-abstractions surrounding the k-th μ-variable in the prefix of the shnf of t_δ. $\mathsf{b}_\mu^{\mathfrak{T}}(\delta)$ is the number of μ-variables in the body part of the shnf of t_δ. $\mathsf{b}_\lambda^{\mathfrak{T}}(\delta, k)$ is the number of term arguments delimited by the k-th μ-variable in the body part of the shnf of t_δ. $\mathsf{b}_\lambda^{\mathfrak{T}}$ corresponds to the function ϕ_t. $\mathsf{Bd}_\mu^{\mathfrak{T}}(\delta, k) = \langle \delta', k'\rangle$ means that the k-th μ-variable in the body part of the shnf of t_δ is bound at k'-th μ-abstraction in the prefix of the shnf of t'_δ.

The conditions are intuitively explained as follows. (1) means that each variable is bound at a position outside of it. (2) and (3) require that, if a variable occurs in a body part of a shnf, it is bound at a prefix which is not in an η-expanded part, which means a part obtained by the η_S-expansion and the fst-reduction. (4) means that, if a λ-variable occurs in an η-expanded part, it is obtained a fst-reduction for a corresponding μ-variable, and all of the nodes below it are in η-expanded parts. (5) means that, if a μ-variable is in an η-expanded part, there is no λ-variable accompanying the μ-variable. (6) means that, if a μ-variable occurs in an η-expanded part, the μ-variables following it are obtained by the η_S-expansion.

Theorem 1. *For $\mathfrak{T} \in \Lambda\mu\text{-}\mathfrak{BT}^+$, there exists a closed $\Lambda\mu$-term t such that $\mathfrak{T} = \mathfrak{BT}_t^+$ iff $\mathfrak{T}$ satisfies all of the conditions in Figure 1.*

Proof. (The detailed proof is in [2].)

Let $\mathfrak{T} = \mathfrak{BT}_t^+$. By definition, when $\mathfrak{T}(\delta) = y$, we have

$$t_\delta \to_h^* \lambda \boldsymbol{x}_\delta^{0,<i_0} \mu\alpha_\delta^0 \cdots \mu\alpha_\delta^{k-2} \lambda x_\delta^{k-1,<i_{k-1}} \mu\alpha_\delta^{k-1}. \\ (y)t^0 \cdots t^{j_0}\alpha_{\delta_0}^{j_0} \cdots \alpha_{\delta_{l-2}}^{j_{l-2}} t^{\omega\cdot(l-1)} \cdots t^{\omega\cdot(l-1)+j_{l-1}-1}\alpha_{\delta_{l-1}}^{j_{l-1}}.$$

Then we define the partial mappings in the condition (0) as follows:

$$\mathsf{p}_\mu^{\mathfrak{T}}(\delta) = k \qquad \mathsf{b}_\mu^{\mathfrak{T}}(\delta) = l$$

$$\mathsf{p}_\lambda^{\mathfrak{T}}(\delta, n) = \begin{cases} i_n & (n < k) \\ 0 & (n \geq k) \end{cases} \qquad \mathsf{b}_\lambda^{\mathfrak{T}}(\delta, n) = \begin{cases} j_n & (n < l) \\ 0 & (n \geq l) \end{cases}$$

$$\mathsf{Bd}_\mu^{\mathfrak{T}}(\delta, n) = \begin{cases} \langle \delta_n, j_n \rangle & (n < l) \\ \langle \delta, n - l + k \rangle & (n \geq l). \end{cases}$$

They are undefined when $\mathfrak{T}(\delta)\uparrow$. They are recursive by definition. It is easy to see that $\mathfrak{T}$ satisfies the conditions (0) through (6).

For the other direction, suppose that $\mathfrak{T} \in \Lambda\mu\text{-}\mathfrak{BT}^+$ satisfies all of the conditions in Figure 1, and we will construct a term $t^{\mathfrak{T}}$ such that $\mathfrak{BT}_{t^{\mathfrak{T}}}^+ = \mathfrak{T}$. In the following, we omit the superscript $\mathfrak{T}$ for each mapping and term.

We have the encodings of elements of ω, ω^2, Δ, BV, and pairs of them in the λ-calculus. These encodings are overlined. By (0), we have λ-representations of $\mathfrak{T}$ and the five partial recursive functions: $\overline{\mathfrak{T}}$, $\overline{\mathsf{p}_\mu}$, $\overline{\mathsf{p}_\lambda}$, $\overline{\mathsf{b}_\mu}$, $\overline{\mathsf{b}_\lambda}$, and $\overline{\mathsf{Bd}_\mu}$. Furthermore we can assume the existence of the following λ-term π:

$$\pi\,\overline{\delta} \to_h^* \begin{cases} \lambda z.z & (\mathfrak{T}(\delta)\downarrow) \\ \text{has no hnf} & (\mathfrak{T}(\delta)\uparrow) \end{cases}$$

We define association lists L_λ and L_μ to map correspondences between actual bound variables and their encodings.

$$\overline{\mathsf{init}}_\lambda = \lambda z.z \qquad [\langle \delta, \mu \rangle \mapsto y]@L_\lambda = \lambda z.(\mathsf{if}\ z = \overline{x_\delta^\mu}\ \mathsf{then}\ y\ \mathsf{else}\ Lz\ \mathsf{fi})$$

$$\overline{\mathsf{init}}_\mu = \lambda z.z \qquad [\langle \delta, k \rangle \mapsto \alpha]@L_\mu = \lambda pz.(\mathsf{if}\ p = \langle \overline{\delta}, \overline{k} \rangle\ \mathsf{then}\ (z)\alpha\ \mathsf{else}\ L_\mu pz\ \mathsf{fi})$$

The term t is recursively defined as follows:

$$
\begin{aligned}
t &= \Theta\overline{[\,]}\,\overline{\mathsf{init}_\lambda}\,\overline{\mathsf{init}_\mu}\\
\Theta\,\overline{\delta}L_\lambda L_\mu &= \pi\,\overline{\delta}\,(F\overline{\delta}\,\overline{0}\,\overline{0}\,(\overline{\mathfrak{T}\,\delta})L_\lambda L_\mu)\\
F\overline{\delta}\,\overline{k}\,\overline{l}VL_\lambda L_\mu &= \begin{cases} G\overline{\delta}\,(\overline{\mathsf{b}_\mu\,\delta})\,\overline{0}VL_\lambda L_\mu & (\mathsf{p}_\mu(\delta)\geq k)\\ \mu\alpha.F\overline{\delta}\,\overline{k+1}\,\overline{0}VL_\lambda([\langle\delta,k\rangle\mapsto\alpha]@L_\mu) & (\mathsf{p}_\lambda(\delta,k)\leq l)\\ \lambda z.F\overline{\delta}\,\overline{k}\,\overline{l+1}V([\langle\delta,\omega\cdot k+l\rangle\mapsto z]@L_\lambda)L_\mu & (\text{otherwise})\end{cases}\\
G\overline{\delta}\,\overline{k}\,\overline{l}VL_\lambda L_\mu &= \begin{cases} L_\lambda V & (k=0\ \&\ l=0)\\ L_\mu(\overline{\mathsf{Bd}_\mu\,\delta\,k-1})(G\overline{\delta}\,\overline{k-1}\,(\overline{\mathsf{b}_\lambda\,\delta\,k-1})VL_\lambda L_\mu) & (k>0\ \&\ l=0)\\ (G\overline{\delta}\,\overline{k}\,\overline{l-1}VL_\lambda L_\mu)(\Theta\overline{\delta::(\omega\cdot k+(l-1))}L_\lambda L_\mu) & (l>0)\end{cases}
\end{aligned}
$$

Then, we can see that $\mathfrak{T}(\delta) = \mathfrak{BT}_t^+(\delta)$ for any $\delta \in \Delta$. □

3.2 Free λ-Variables

The discussion in the previous subsection can be extended to $\Lambda\mu$-terms with free λ-variables. We suppose the set of free λ-variables FV_λ, which is disjoint from BV_λ. The codomain of $\Lambda\mu$-$\mathfrak{BT}^+$ is extended to $BV_\lambda \cup FV_\lambda \cup \{\bot\}$. We define $FV_\lambda(\mathfrak{T}) = \{z \in FV_\lambda \mid \mathfrak{T}(\delta) = z \text{ for some } \delta\}$, and we require the following additional condition.

(7) $\#FV_\lambda(\mathfrak{T}) < \omega$

Then, the encoding of the variables are extended to

$$
\overline{y} = \begin{cases} \overline{y} & (y \in BV_\lambda)\\ y & (y \in FV_\lambda(\mathfrak{T})),\end{cases}
$$

which can be defined due to the condition (7). Notice that for any association list L_λ of λ-variables and $y \in FV_\lambda$, we have $L_\lambda\, y \to_h^* \overline{\mathsf{init}}_\lambda\, y \to_h y$.

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References

1. Nakajima, R. Infinite normal forms for the λ-calculus. In *Symposium on Lambda-Calculus and Computer Science Theory*, pages 62–82, 1975.
2. Koji Nakazawa. Characterizing trees for Lambda-mu terms. Extended version of this paper with the proof of the main theorem. Available at http://www.sqlab.i.is.nagoya-u.ac.jp/~nakazawa/papers/hor2016appendix.pdf.
3. A. Saurin. Standardization and Böhm trees for Λμ-calculus. In Blume, M., Kobayashi, N., and Vidal, G., editors, *Tenth International Symposium on Functional and Logic Programming (FLOPS 2010)*, volume 6009 of *LNCS*, pages 134–149. Springer, 2010.